Matematika 2, vježbe, 29.4.2020.

* DIREKTNA INTEGRACIJA - nastavak

* Kod računanja integrala često vam je korisno sljedeće pravilo (vidi udžbenik, str. 258)

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$$

derivacija desne strane upravo daje (provjerite)

Primjenu prethodnog pravila ilustrirat ćemo u sljedećem zadatku:

Zad Izračunajte sljedeće integrale:

a) $$\int \underbrace{\frac{5x^4+6x}{x^5+3x^2+6}}_{\frac{f'(x)}{f(x)}}\,dx = \ln|x^5+3x^2+6| + C$$

b) $$\int \underbrace{\frac{6x^5+5-24x^2}{x^6+5x-8x^3}}_{\frac{f'(x)}{f(x)}}\,dx = \ln|x^6+5x-8x^3| + C$$

c) $$\int \frac{6+48x}{3x+12x^2}\,dx = 2\int \frac{3+24x}{3x+12x^2}\,dx = 2\ln|3x+12x^2| + C$$

d) $$\int \frac{-15x^4+36x}{x^5-6x^2}\,dx = -3\int \frac{5x^4-12x}{x^5-6x^2}\,dx = -3\ln|x^5-6x^2| + C$$

* Metoda SUPSTITUCIJE *

Zad Izračunajte sljedeće integrale:

a) $$\int (x-5)^5 dx = \left| \begin{matrix} x-5=t \\ 1\cdot dx = 1\cdot dt \\ (dx = dt) \end{matrix} \right| = \int t^5 dt = \frac{t^6}{6} + C$$

$$= \frac{(x-5)^6}{6} + C$$

b) $$\int (2x-6)^7 dx = \left| \begin{matrix} 2x-6=t \\ 2dx = dt \ /:2 \\ dx = \frac{dt}{2} \end{matrix} \right| = \int t^7 \frac{dt}{2} = \frac{1}{2}\int t^7 dt$$

$$= \frac{1}{2}\frac{t^8}{8} + C = \frac{t^8}{16} + C = \frac{(2x-6)^8}{16} + C$$

c) $$\int \sqrt[3]{x-2}\, dx = \int (x-2)^{\frac{1}{3}} dx = \left| \begin{matrix} x-2=t \\ dx = dt \end{matrix} \right|$$

$$= \int t^{\frac{1}{3}} dt = \frac{t^{\frac{1}{3}+1}}{\frac{1}{3}+1} + C = \frac{3}{4} t^{\frac{4}{3}} + C = \frac{3}{4}(x-2)^{\frac{4}{3}} + C$$

d) $$\int (x^2+3)^2 \cdot x\, dx = \left| \begin{matrix} x^2+3=t \\ 2x\,dx = dt \ /:2x \\ dx = \frac{dt}{2x} \end{matrix} \right| = \int t^2 \cdot \cancel{x} \frac{dt}{2\cancel{x}}$$

$$= \frac{1}{2}\int t^2 dt = \frac{1}{2}\frac{t^3}{3} + C = \frac{t^3}{6} + C = \frac{(x^2+3)^3}{6} + C$$

e) $$\int \frac{x^3}{\sqrt[3]{2x^4+5}} dx = \left| \begin{matrix} 2x^4+5=t \\ 8x^3 dx = dt \ /:8x^3 \\ dx = \frac{dt}{8x^3} \end{matrix} \right| = \int \frac{\cancel{x^3}}{\sqrt[3]{t}} \frac{dt}{8\cancel{x^3}}$$

$$= \frac{1}{8}\int \frac{1}{\sqrt[3]{t}} dt = \frac{1}{8}\int t^{-\frac{1}{3}} dt = \frac{1}{8}\frac{t^{\frac{2}{3}}}{\frac{2}{3}} + C = \frac{1}{8}\cdot\frac{3}{2} t^{\frac{2}{3}} + C = \frac{3}{16}(2x^4+5)^{\frac{2}{3}} + C$$

③

f) $\int \frac{x^4}{\sqrt[4]{3x^5+2}}\,dx = \left| \begin{array}{l} 3x^5+2=t \\ 15x^4\,dx = dt \;/ : 15x^4 \\ dx = \frac{dt}{15x^4} \end{array} \right|$

$= \int \frac{\cancel{x^4}}{\sqrt[4]{t}} \frac{dt}{15\cancel{x^4}} = \frac{1}{15}\int \frac{1}{\sqrt[4]{t}}\,dt = \frac{1}{15}\int t^{-\frac{1}{4}}\,dt$

$= \frac{1}{15}\frac{t^{\frac{3}{4}}}{\frac{3}{4}} + c = \frac{1}{15}\cdot\frac{4}{3}\,t^{\frac{3}{4}} + c = \frac{4}{45}(3x^5+2)^{\frac{3}{4}} + c$

g) $\int \cos(4x-1)\,dx = \left| \begin{array}{l} 4x-1=t \\ 4dx = dt \;/ : 4 \\ dx = \frac{dt}{4} \end{array} \right| = \int \cos t \,\frac{dt}{4}$

$= \frac{1}{4}\int \cos t\,dt = \frac{1}{4}\sin t + c = \frac{1}{4}\sin(4x-1) + c$

h) $\int 3\sin(2x+5)\,dx = 3\int \sin(2x+5)\,dx = \left| \begin{array}{l} 2x+5=t \\ 2dx = dt \;/ : 2 \\ dx = \frac{dt}{2} \end{array} \right|$

$= 3\cdot\int \sin t\,\frac{dt}{2} = \frac{3}{2}\int \sin t\,dt = \frac{3}{2}(-\cos t) + c$

$= -\frac{3}{2}\cos(2x+5) + c$

i) $\int \frac{\cos x}{\sqrt{1+\sin x}}\,dx = \left| \begin{array}{l} 1+\sin x = t \\ \cos x\,dx = dt \;/ : \cos x \\ dx = \frac{dt}{\cos x} \end{array} \right| = \int \frac{\cancel{\cos x}}{\sqrt{t}} \frac{dt}{\cancel{\cos x}}$

$= \int \frac{dt}{\sqrt{t}} = \int t^{-\frac{1}{2}}\,dt = \frac{t^{\frac{1}{2}}}{\frac{1}{2}} + c = 2t^{\frac{1}{2}} + c$

$= 2\sqrt{1+\sin x} + c$

j) $\int \frac{x}{(1+x^2)^3}\,dx = \left| \begin{array}{l} 1+x^2=t \\ 2x\,dx = dt \quad /:2x \\ dx = \frac{dt}{2x} \end{array} \right|$

$= \int \frac{\cancel{x}}{t^3} \frac{dt}{2\cancel{x}} = \frac{1}{2}\int t^{-3}\,dt = \frac{1}{2}\frac{t^{-2}}{-2} + C = -\frac{1}{4}t^{-2} + C$

$= -\frac{1}{4}(1+x^2)^{-2} + C$

k) $\int \frac{\sqrt{1+\ln x}}{x}\,dx = \left| \begin{array}{l} 1+\ln x = t \\ \frac{1}{x}\,dx = dt \quad /\cdot x \\ dx = x\cdot dt \end{array} \right|$

$= \int \frac{\sqrt{t}}{\cancel{x}}\cdot \cancel{x}\,dt = \int t^{\frac{1}{2}}\,dt = \frac{t^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{2}{3}(1+\ln x)^{\frac{3}{2}} + C$

l) $\int \frac{\ln(2x+2)}{x+1}\,dx = \left| \begin{array}{l} \ln(2x+2) = t \\ \underbrace{\frac{2}{2x+2}\,dx = dt} \\ \frac{\cancel{2}}{\cancel{2}(x+1)}\,dx = dt \quad /\cdot(x+1) \\ dx = (x+1)\,dt \end{array} \right|$

$= \int \frac{t}{\cancel{x+1}}\cancel{(x+1)}\,dt = \int t\,dt = \frac{t^2}{2} + C = \frac{\ln^2(2x+2)}{2} + C$

$= \frac{(\ln(2x+2))^2}{2} + C$

lj) $\int \frac{\ln(7x-7)}{5x-5}\,dx = \frac{1}{5}\int \frac{\ln(7x-7)}{x-1}\,dx = \left| \begin{array}{l} \ln(7x-7) = t \\ \frac{7}{7x-7}\,dx = dt \\ \frac{1}{x-1}\,dx = dt \Rightarrow dx = (x-1)\,dt \end{array} \right|$

$= \frac{1}{5}\int \frac{t}{\cancel{x-1}}\cancel{(x-1)}\,dt = \frac{1}{5}\int t\,dt = \frac{1}{5}\frac{t^2}{2} + C = \frac{t^2}{10} + C$

$= \frac{(\ln(7x-7))^2}{10} + C.$

Metoda PARCIJALNE INTEGRACIJE

$$\int u\,dv = u \cdot v - \int v\,du$$

u polaznom integralu nešto odaberemo za u, a sve što je preostalo odaberemo za dv

↳ s desne strane jednakosti nam osim u treba i du : du dobijemo iz u tj. du je derivacija od u

treba nam i v : v ćemo dobiti iz dv tako da INTEGRIRAMO dv.

Zad Metodom parcijalne integracije izračunajte integrale:

a) $\int x \cdot e^x\,dx =$

Kako odabrati $\boxed{u \text{ i } dv}$?

npr. $u = e^x$, a sve što je preostalo rekli smo da ćemo odabrati za dv, tj. ovdje je $dv = x\,dx$

du dobijemo tako da deriviramo u:

$\boxed{du = e^x\,dx}$

v dobijemo tako da integriramo dv:

$\int 1\,dv = \int x\,dx$

$v = \frac{x^2}{2}$

Sada prema formuli za parcijalnu integr.:

$$\int x e^x\,dx = \underbrace{e^x}_{u} \cdot \underbrace{\frac{x^2}{2}}_{v} - \int \underbrace{\frac{x^2}{2}}_{v} \cdot \underbrace{e^x\,dx}_{du}$$

← još kompliciranije!

nije dobar odabir u i dv!

Kako "dobro" odabrati? Primijetimo da smo na prethodnoj str. za dv odabrali x dx a kada smo integrirali x stupanj mu se povećao! Da smo derivirali x stupanj bi mu se smanjio, tj. zadatak bi se pojednostavio! Dakle, uvijek kada imamo neki polinom njega je dobro uzeti za u. Idemo ponovo rješavati:

$$\int \underbrace{x}_{u} \underbrace{e^x dx}_{dv} = \left| \begin{array}{l} u = x \Rightarrow du = dx \\ dv = e^x dx \Rightarrow \int dv = \int e^x dx \Rightarrow v = e^x \end{array} \right|$$

$$= x \cdot e^x - \int e^x dx = \underline{\underline{x \cdot e^x - e^x + C}}$$

b) $$\int \underbrace{(6x+2)}_{u} \cdot \underbrace{\sin x \, dx}_{dv} = \left| \begin{array}{l} u = 6x+2 \Rightarrow du = 6 dx \\ dv = \sin x \, dx \Rightarrow \int dv = \int \sin x \, dx \Rightarrow v = -\cos x \end{array} \right|$$

$$= (6x+2)(-\cos x) - \int (-\cos x) \cdot 6 \, dx$$

$$= -(6x+2)\cos x + 6 \int \cos x \, dx = -(6x+2)\cos x + 6 \sin x + C$$

c) $$\int (x^2+1) \cdot 3^x dx = \left| \begin{array}{l} u = x^2+1 \Rightarrow du = 2x dx \\ dv = 3^x dx \Rightarrow v = \frac{3^x}{\ln 3} \end{array} \right|$$

$$= (x^2+1) \cdot \frac{3^x}{\ln 3} - \int \frac{3^x}{\ln 3} \cdot 2x \, dx = (x^2+1) \cdot \frac{3^x}{\ln 3} - \frac{2}{\ln 3} \underbrace{\int x \cdot 3^x dx}_{} \quad (**)$$

Izračunajmo integral koji je preostao, ponovo pomoću parcijalne integracije:

$$\int x \cdot 3^x dx = \left| \begin{array}{l} u = x \Rightarrow du = dx \\ dv = 3^x dx \Rightarrow v = \frac{3^x}{\ln 3} \end{array} \right| = x \cdot \frac{3^x}{\ln 3} - \int \frac{3^x}{\ln 3} dx$$

$$= x \cdot \frac{3^x}{\ln 3} - \frac{1}{\ln 3} \int 3^x dx$$

$$= \frac{x \cdot 3^x}{\ln 3} - \frac{1}{\ln 3} \cdot \frac{3^x}{\ln 3} = \frac{x \cdot 3^x}{\ln 3} - \frac{3^x}{(\ln 3)^2}$$

uvrstimo to rješenje u (**) i sada je konačno rj:

konačno: $$(x^2+1)\frac{3^x}{\ln 3} - \frac{2}{\ln 3}\left(\frac{x \cdot 3^x}{\ln 3} - \frac{3^x}{(\ln 3)^2}\right) + c$$

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d) $$\int (x^2-2)\cos x \, dx = \left| \begin{array}{l} u = x^2-2 \Rightarrow du = 2x\,dx \\ dv = \cos x\,dx \Rightarrow v = \sin x \end{array} \right|$$

$$= (x^2-2)\sin x - \int \sin x \cdot 2x\,dx = (x^2-2)\sin x - 2\underbrace{\int x \cdot \sin x\,dx}\ (**)$$

Riješimo integral $\int x \sin x\,dx = \left| \begin{array}{l} u = x \Rightarrow du = dx \\ dv = \sin x\,dx \Rightarrow v = -\cos x \end{array} \right|$

$$= x \cdot (-\cos x) - \int (-\cos x)\,dx = -x\cos x + \int \cos x\,dx$$

$$= \underline{-x\cos x + \sin x}$$ uvrstimo to rješenje u (**)

i sada je konačno rj:

$$= (x^2-2)\sin x - 2\left(-x\cos x + \sin x\right) + c$$

e) $\int \frac{\ln x}{\sqrt[5]{x^3}}\,dx = \int x^{-\frac{3}{5}} \ln x\, dx$ (8)

$\to$ kolokvij 2017.

$\hookrightarrow$ ne možemo ga odabrati za dv jer dv treba integrirati da bi dobili v, a mi $\ln x$ ne znamo integrirati – znamo ga samo derivirati, zato ćemo ga odabrati za u.

$$= \left| \begin{array}{l} u = \ln x \Rightarrow du = \frac{1}{x}\,dx \\ dv = x^{-\frac{3}{5}}\,dx \Rightarrow v = \frac{5}{2}x^{\frac{2}{5}} \end{array} \right|$$

$$= \ln x \cdot \frac{5}{2}x^{\frac{2}{5}} - \int \frac{5}{2} x^{\frac{2}{5}} \cdot \frac{1}{x}\,dx$$

$$= x^{\frac{2}{5}-1} = x^{\frac{2-5}{5}} = x^{-\frac{3}{5}}$$

$$= \frac{5}{2}\ln x \cdot x^{\frac{2}{5}} - \frac{5}{2}\int x^{-\frac{3}{5}}\,dx = \frac{5}{2}\ln x \cdot x^{\frac{2}{5}} - \frac{5}{2}\cdot x^{\frac{2}{5}}\cdot\frac{5}{2} + c$$

$$= \frac{5}{2}x^{\frac{2}{5}}\cdot \ln x - \frac{25}{4}x^{\frac{2}{5}} + c$$

f) $\int \frac{\ln x}{\sqrt[4]{x}}\,dx = \int x^{-\frac{1}{4}} \ln x\,dx = \left| \begin{array}{l} u = \ln x \Rightarrow du = \frac{1}{x}\,dx \\ dv = x^{-\frac{1}{4}}\,dx \Rightarrow v = \frac{4}{3}x^{\frac{3}{4}} \end{array} \right|$

$$= \ln x \cdot \frac{4}{3}x^{\frac{3}{4}} - \int \frac{4}{3} x^{\frac{3}{4}} \cdot \frac{1}{x}\,dx$$

$$x^{\frac{3}{4}-1} = x^{-\frac{1}{4}}$$

$$= \frac{4}{3}x^{\frac{3}{4}}\ln x - \frac{4}{3}\int x^{-\frac{1}{4}}\,dx = \frac{4}{3}x^{\frac{3}{4}}\ln x - \frac{4}{3}\frac{x^{\frac{3}{4}}}{\frac{3}{4}} + c$$

$$= \frac{4}{3}x^{\frac{3}{4}}\ln x - \frac{16}{9}x^{\frac{3}{4}} + c$$